Mathematics 181A **Fall 2010**

Exam 2

October 7, 2010

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Name

Technology used: ____________________________________ **Only write on one side of each page.**
Show all of your work. Calculators may be used for numerical calculations and answer checking only.

1. $[10, 10, 10$ points$]$ Evaluate the following integrals. Show all of your work.

1. $\int \cos^5(3x)\ dx = \int \left[\cos^2(3x)\right]^2 \cos(3x)\ dx = \frac{1}{3}\int \left(1-\sin^2(3x)\right)^2 d(\sin 3x) = \frac{1}{3}\int \left(1-2u^2+u^4\right) du = \frac{1}{3}u - \frac{2}{9}u^3 + \frac{1}{15}u^5 + C$. Now backsubstitute $u = \sin(3x)$.
2. $\int \sec^4(2x)\ \ dx = \int \left[\sec^2(2x)\right]\sec^2(2x)\ \ dx = \frac{1}{2}\int \left(1+\tan^2 2x\right) d(\tan 2x) = \frac{1}{2}\int \left(1+u^2\right) du = \frac{1}{6}u^3 + \frac{1}{2}u + C$. Now backsubstitute $u = \tan(2x)$.
3. $\int y \ln(y)\ \ dy = \frac{1}{2}y^2 \ln(y) - \int \frac{1}{2}y^2 \frac{1}{y} dy = \frac{1}{2}y^2 \ln(y) - \int \frac{1}{2}y\ dy = \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + C$
   (a) Where we used integration by parts and $u = \ln(y)$, $dv = y$, $du = \frac{1}{y}dy$, $v = \frac{1}{2}y^2$ we

2. $[15$ points$]$ Find the length of the curve $y = x^{1/2} - (1/3)\,x^{3/2}$, $1 \le x \le 4$.

1. Set $x = t$ and $y = t^{1/2} - (1/3)\,t^{3/2}$, then $\left[\frac{dx}{dt}\right]^2 = [1]^2 = 1$ and $\left[\frac{dy}{dt}\right]^2 = \left[\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\right]^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x$
2. So

$$\begin{aligned} ds &= \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt = \sqrt{1 + \left(\frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x\right)} dt \\ &= \sqrt{\frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x} dt = \sqrt{\left(\left[\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right]^2\right)} dt = \left|\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right| dt \end{aligned}$$

3. So $s = \int_1^4 \left|\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right| dt = \int \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)\ dt = x^{1/2} + \frac{1}{3}x^{3/2}\Big]_1^4 = \frac{10}{3}$

3. $[15$ points$]$ Find the area of the surface generated by revolving the curve $y = \sqrt{4x - x^2}$, $1 \le x \le 2$ about the x-axis.

1. Set $x = t$ and $y = \left(4t - t^2\right)^{1/2}$ so that

$$\begin{aligned} \left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 &= 1 + \left[\frac{\frac{1}{2}(4-2t)}{\sqrt{4t-t^2}}\right]^2 = 1 + \frac{(2-t)^2}{4t-t^2} \\ &= \frac{4t - t^2 + (2-t)^2}{4t-t^2} = \frac{4}{4t-t^2} \end{aligned}$$

2. So, the surface area is $2\pi\int_1^2 (radius)\,ds = 2\pi\int_1^2 \sqrt{4t-t^2}\sqrt{\frac{4}{4t-t^2}}\ dt = 2\pi\int_1^2 2\ dt = 4\pi$.

4. $[15$ points$]$ Solve the initial value problem $\frac{dy}{dx} = \frac{y\ln(y)}{1+x^2}$, $y(0) = e^2$.

1. Separate variables to obtain $\int \frac{1}{y\ln(y)}\frac{dy}{dx}\ dx = \int \frac{1}{1+x^2}\ dx$ and use the substitution $u = \ln(y)$, $du = \frac{1}{y}dy$ on the left integral.
2. $\int \frac{1}{u}\ du = \ln|u| + C_1 = \ln|\ln y| + C_1 = \arctan(x) + C_2$. Setting $C = C_2 - C_1$ we get
3. $\ln|\ln y| = \arctan(x) + C$ and the initial condition tells us that $\ln\left|\ln\left(e^2\right)\right| = \arctan(0) + C$ so $C = \ln\left(\ln\left(e^2\right)\right) = \ln(2)$
4. So $\ln|\ln y| = \arctan(x) + \ln(2)$ which implies

$$\begin{aligned} \ln y &= e^{\arctan(x)+\ln(2)} = e^{a\arctan(x)} \cdot e^{\ln(2)} \\ &= 2e^{\arctan(x)} \\ \text{So, } y &= e^{2e^{\arctan(x)}} \end{aligned}$$

6. [10 points each] A deep dish-apple pie, whose internal temperature was 220°F when removed from the oven was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How much longer did it take for the pie to cool to 70°F?

1. Using $T(t) - A = (T_0 - A)e^{-kt}$ with $A = 40$ and $T_0 = 220$ and $T(15) = 180$ we get

$$\begin{aligned} 180 - 40 &= (220 - 40)e^{-k(15)} \\ \frac{\ln\left(\frac{7}{9}\right)}{-15} &= k \end{aligned}$$

2. Then using this k and solving for t in

$$\begin{aligned} 70 - 40 &= (220 - 40)e^{-kt} \\ \ln\left(\frac{1}{6}\right) &= -kt \\ t &= -\ln\left(\frac{1}{6}\right) / \frac{\ln\left(\frac{7}{9}\right)}{-15} \\ &\approx 106.9 \text{ minutes} \end{aligned}$$

3. The answer is $106.9 - 15 = 91.9$ minutes.

7. [15 points] A disk of radius 2 is revolved around the y-axis to form a solid sphere. A round hole of radius $\sqrt{3}$, centered on the y-axis is bored through the sphere. Find the volume of material removed from the sphere.

1. Using cylindrical shells we see the volume removed from the sphere is $2\pi\int_0^{\sqrt{3}} x\sqrt{4-x^2}\ dx$ which we can integrate using $u = 4 - x^2$, $du = -2x\ dx$. The removed volume is $2\pi\int_0^{\sqrt{3}} x\sqrt{4-x^2}\ dx = \frac{14}{3}\pi$

Extra Credit [5 points] At each point on the curve $y = 2\sqrt{x}$, a line segment of length $h = y$ is drawn perpendicular to the xy-plane. Set up an integral that equals the area of the surface formed by these perpendiculars from $x = 0$ to $x = 3$. [Note that this is **not** a surface of revolution so none of the formulas in Chapter 6 apply. Develop your own integral by using Riemann sums to estimate the area of the surface.]

1. The surface extends vertically upward from the **curve** $y = 2\sqrt{x}$. If we partition the graph of $y = 2\sqrt{x}$ into many small arcs of length approximately Δs_k, then the area of the surface above the kth arc is approximately $2\sqrt{x_k}\ \Delta s_k$. Thus the associated Riemann sum that approximates

the total area is $\sum_{k=1}^{n} 2\sqrt{x_k}\ \Delta s_k$ and since $f(x) = 2\sqrt{x}$ is a smooth curve on the given domain we know that the limit of Riemann sums exists and is equal to the integral $\int_0^3 2\sqrt{x}\ ds$. To compute this actual area, we need to compute $ds = \sqrt{1 + x^{-1}} dx = \frac{x^{1/2}}{\sqrt{x+1}}$so the integral is

$$\int_0^3 2x^{1/2} \cdot \frac{x^{1/2}}{(x+1)^{1/2}}\ dx = 2\int_0^3 \frac{x}{(x+1)^{1/2}}\ dx$$

which, when integrated by using the "Rule of Thumb" substitution $u = x + 1$, yields a value of $\frac{16}{3}$.